Available online at www.sciencedirect.com
ScienceDirect
Computers and Mathematics with Applications 55 (2008) 2999–3002
An International Journal
computers & mathematics with applications
www.elsevier.com/locate/camwa

Erratum

# Erratum to "Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces" [Comput. Math. Appl. 54 (2007) 872–877]

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Received 5 October 2007; accepted 17 November 2007

**Abstract**

We show strong convergence for Mann and Ishikawa iterates of multivalued nonexpansive mapping $T$ under some appropriate conditions, which revises a gap in Panyanak [B. Panyanak, Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces, Comput. Math. Appl. 54 (2007) 872–877]. Furthermore, we also give an affirmative answer to Panyanak's open question.


*Keywords:* Ishikawa iterates; Strong convergence; Uniformly convex Banach spaces

Let $E$ be a Banach space and $K$ a nonempty subset of $E$. We shall denote $CB(E)$ by the family of nonempty closed and bounded subsets of $E$ and the family of nonempty bounded proximinal subsets of $E$ (see [1]). Let $H$ be the Hausdorff metric on $CB(E)$, that is,

$$H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{x\in B} d(x, A)\} \quad \text{for any } A, B \in CB(E),$$

where $d(x, B) = \inf\{\|x - y\|; y \in B\}$. A multivalued mapping $T : K \to CB(E)$ is said to be *nonexpansive*, if for any $x, y \in K$, such that $H(Tx, Ty) \le \|x - y\|$. A point $x$ is called a fixed point of $T$ if $x \in Tx$. From now on, $F(T)$ stands for the fixed point set of a mapping $T$.

Recently, Panyanak [1] introduced the following Ishikawa iterates of a multivalued mapping $T$. Let $K$ be a nonempty convex subset of $E$, fix $p \in F(T)$ and $x_0 \in K$,

$$y_n = (1 - \beta_n)x_n + \beta_n z_n, \quad \beta_n \in [0, 1], n \ge 0,$$

where $z_n \in Tx_n$ such that $\|z_n - p\| = d(p, Tx_n)$, and

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n z'_n, \quad \alpha_n \in [0, 1], n \ge 0,$$

DOI of original article: 10.1016/j.camwa.2007.03.012.
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doi:10.1016/j.camwa.2007.11.042

where $z_n' \in Ty_n$ such that $\|z_n' - p\| = d(p, Ty_n)$. It is obvious that $x_n$ depends on $p$ and $T$. For $p \in F(T)$, we have

$$\|z_n - p\| = d(p, Tx_n) \leq H(Tp, Tx_n) \leq \|x_n - p\|$$

and

$$\|z_n' - p\| = d(p, Ty_n) \leq H(Tp, Ty_n) \leq \|y_n - p\|.$$

Clearly, if $q \in F(T)$ and $q \neq p$, then the above inequalities cannot be assured. Namely, from the monotony of $\{\|x_n - p\|\}$ in the proof of [1, Theorem 3.1], we cannot obtain $\{\|x_n - q\|\}$ is a decreasing sequence. Hence, the conclusion of Theorem 3.1 in [1] cannot be reached.

Motivated by solving the above gap, we have tried to modify it. The aim of this paper is to find an iteration instead of the above one and to overcome its limitation. We will construct the following iteration.

Let $K$ be a nonempty convex subset of $E$, $\beta_n \in [0, 1]$, $\alpha_n \in [0, 1]$ and $\gamma_n \in (0, +\infty)$ such that $\lim_{n\to\infty} \gamma_n = 0$. Choose $x_0 \in K$ and $z_0 \in Tx_0$. Let

$$y_0 = (1 - \beta_0)x_0 + \beta_0 z_0.$$

There exists $z_0' \in Ty_0$ such that $\|z_0 - z_0'\| \leq H(Tx_0, Ty_0) + \gamma_0$ (see [2,3]). Let

$$x_1 = (1 - \alpha_0)x_0 + \alpha_0 z_0'.$$

There is $z_1 \in Tx_1$ such that $\|z_1 - z_0'\| \leq H(Tx_1, Ty_0) + \gamma_1$. Take

$$y_1 = (1 - \beta_1)x_1 + \beta_1 z_1.$$

There exists $z_1' \in Ty_1$ such that $\|z_1 - z_1'\| \leq H(Tx_1, Ty_1) + \gamma_1$. Let

$$x_2 = (1 - \alpha_1)x_1 + \alpha_1 z_1'.$$

Inductively, we have

$$\begin{aligned} y_n &= (1 - \beta_n)x_n + \beta_n z_n, \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n z_n', \end{aligned} \tag{1}$$

where $\|z_n - z_n'\| \leq H(Tx_n, Ty_n) + \gamma_n$ and $\|z_{n+1} - z_n'\| \leq H(Tx_{n+1}, Ty_n) + \gamma_n$ for $z_n \in Tx_n$ and $z_n' \in Ty_n$.

We now show the strong convergence of the Ishikawa iteration (1) which shakes off the objection in [1, Theorem 3.1].

**Theorem 1.** *Let $K$ be a nonempty compact convex subset of a uniformly convex Banach space $E$. Suppose that $T : K \to CB(K)$ is a multivalued nonexpansive mapping and $F(T) \neq \emptyset$ satisfying $T(y) = \{y\}$ for any fixed point $y \in F(T)$.*

*Let $\{x_n\}$ be the sequence of Ishikawa iterates defined by (1). Assume that*

(i) $\alpha_n, \beta_n \in [0, 1)$; (ii) $\lim_{n\to\infty} \beta_n = 0$ *and* (iii) $\sum_{n=0}^{\infty} \alpha_n \beta_n = \infty$.

*Then as $n \to \infty$, the sequence $\{x_n\}$ strongly converges to some fixed point of $T$.*

**Proof.** Take $p \in F(T)$ (noting $Tp = \{p\}$ and $\|z_n - p\| = d(z_n, Tp)$). Using a similar proof of Theorem 3.1 as in [1] (Xu's inequality, see[1, Lemma 2.3]), we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n \|z_n' - p\|^2 - \alpha_n(1 - \alpha_n)\varphi(\|x_n - z_n'\|) \\ &\leq (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n (H(Ty_n, Tp))^2 \\ &\leq (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n \|y_n - p\|^2 \\ &\leq (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n[(1 - \beta_n)\|x_n - p\|^2 + \beta_n\|z_n - p\|^2 - \beta_n(1 - \beta_n)\varphi(\|x_n - z_n\|)] \\ &\leq (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n[(1 - \beta_n)\|x_n - p\|^2 \\ &\quad + \beta_n(H(Tx_n, Tp))^2 - \beta_n(1 - \beta_n)\varphi(\|x_n - z_n\|)] \\ &\leq \|x_n - p\|^2 - \alpha_n\beta_n(1 - \beta_n)\varphi(\|x_n - z_n\|). \end{aligned}$$

Therefore,

$$\|x_{n+1} - p\|^2 \le \|x_n - p\|^2$$

and

$$\alpha_n \beta_n (1 - \beta_n)\varphi(\|x_n - z_n\|) \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2. \tag{2}$$

Then $\{\|x_n - p\|\}$ is a decreasing sequence and further $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in F(T)$. It follows from (2) that

$$\sum_{n=0}^{\infty} \alpha_n \beta_n (1 - \beta_n)\varphi(\|x_n - z_n\|) \le \|x_1 - p\|^2.$$

The remainder proof is the same as Theorem 3.1 of [1], we omit it.

A multivalued mapping $T : K \to CB(K)$ is said to satisfy *Condition* I if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for $r \in (0, \infty)$ such that

$$d(x, Tx) \ge f(d(x, F(T))) \quad \text{for all } x \in K. \quad \square$$

Next we give the affirmative answer of the open question in [1] using the iteration (1).

**Theorem 2.** *Let $K$ be a nonempty closed convex subset of a uniformly convex Banach space $E$. Suppose that $T : K \to CB(K)$ is a multivalued nonexpansive mapping that satisfies Condition* I. *Let $\{x_n\}$ be the sequence of Ishikawa iterates defined by* (1). *Assume that $F(T) \ne \emptyset$ satisfying $T(y) = \{y\}$ for any fixed point $y \in F(T)$ and $\alpha_n, \beta_n \in [a, b] \subset (0, 1)$. Then as $n \to \infty$, the sequence $\{x_n\}$ strongly converges to some fixed point of $T$.*

**Proof.** Using a similar proof of Theorem 1, we obtain $\lim_{n\to\infty} \|x_n - p\|^2$ exists for $p \in F(T)$ and

$$\alpha_n \beta_n (1 - \beta_n)\varphi(\|x_n - z_n\|) \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

Then

$$a^2(1 - b)\varphi(\|x_n - z_n\|) \le \alpha_n \beta_n (1 - \beta_n)\varphi(\|x_n - z_n\|) \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

Thus, $\lim_{n\to\infty} \varphi(\|x_n - z_n\|) = 0$ and hence $\lim_{n\to\infty} \|x_n - z_n\| = 0$. As $z_n \in Tx_n$, then

$$d(x_n, Tx_n) \le \|x_n - z_n\|.$$

Therefore, $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Furthermore Condition I implies

$$\lim_{n\to\infty} d(x_n, F(T)) = 0. \quad \square$$

The remainder of the proof is the same as Theorem 3.8 of [1], and we omit it.

**Remark.** The above results holds for Mann iteration ($\beta_n \equiv 0$ in (1)). For the conclusion of Theorem 2, let $x_{n+1} = (1 - \alpha_n)x_n + \alpha_n z_n$ for $z_n \in Tx_n$ and $\alpha_n \in [a, b] \subset (0, 1)$. Then we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n \|z_n - p\|^2 - \alpha_n(1 - \alpha_n)\varphi(\|x_n - z_n\|) \\
&\le (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n (H(Tx_n, Tp))^2 - \alpha_n(1 - \alpha_n)\varphi(\|x_n - z_n\|) \\
&\le (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n \|x_n - p\|^2 - \alpha_n(1 - \alpha_n)\varphi(\|x_n - z_n\|) \\
&\le \|x_n - p\|^2 - \alpha_n(1 - \alpha_n)\varphi(\|x_n - z_n\|).
\end{aligned}$$

Hence,

$$a(1 - b)\varphi(\|x_n - z_n\|) \le \alpha_n(1 - \beta_n)\varphi(\|x_n - z_n\|) \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

Thus we also have $\lim_{n\to\infty} \varphi(\|x_n - z_n\|) = 0$ and hence

$$\lim_{n\to\infty} d(x_n, F(T)) = 0.$$

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